

# Compactness of $\omega^\lambda$ for $\lambda$ singular

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*Abstract:* We characterize the compactness properties of the product of  $\lambda$  copies of the space  $\omega$  with the discrete topology, dealing in particular with the case  $\lambda$  singular, using regular and uniform ultrafilters, infinitary languages and nonstandard elements. We also deal with products of uncountable regular cardinals with the order topology.

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The problem of determining the compactness properties satisfied by powers of the countably infinite discrete topological space  $\omega$  originates from Stone [17], who proved that  $\omega^{\omega_1}$  is not normal, hence, in particular, not Lindelöf. More generally, Mycielski [16] showed that  $\omega^\kappa$  is not finally  $\kappa$ -compact, for every infinite cardinal  $\kappa$  strictly less than the first weakly inaccessible cardinal. Recall that a topological space is said to be *finally  $\kappa$ -compact* if any open cover has a subcover of cardinality strictly less than  $\kappa$ . *Lindelöfness* is the same as final  $\omega_1$ -compactness. Previous work on the subject had been also done by A. Ehrenfeucht, P. Erdős, A. Hajnal and J. Łoś; see [16] for details. Mycielski's result cannot be generalized to arbitrarily large cardinals: if  $\kappa$  is weakly compact then  $\omega^\kappa$  is indeed finally  $\kappa$ -compact: see Keisler and Tarski [8, Theorem 4.32]. Related work is due to D. V. Čudnovskiĭ, W. Hanf, D. Monk, D. Scott, S. Todorčević and S. Ulam, among many others. With regard to powers of  $\omega$  a more refined result has been obtained by Mrówka who, e. g., in [15] showed that if the infinitary language  $\mathcal{L}_{\omega_1, \omega}$  is  $(\kappa, \kappa)$ -compact, then  $\omega^\kappa$  is finally  $\kappa$ -compact. This is a stronger result since Boos [2] showed that it is possible that  $\mathcal{L}_{\omega_1, \omega}$  is  $(\kappa, \kappa)$ -compact, even, that  $\mathcal{L}_{\kappa, \omega}$  is  $(\kappa, \kappa)$ -compact, without  $\kappa$  being weakly compact. To the best of our knowledge the gap between Mycielski's and Mrówka's results had not been exactly filled until we showed in [14] that Mrówka gives the optimal bound, that is, for  $\kappa$  regular,  $\omega^\kappa$  is finally  $\kappa$ -compact if and only if  $\mathcal{L}_{\omega_1, \omega}$  is  $(\kappa, \kappa)$ -compact. The aim of the present note is to show that the result holds also for a singular cardinal  $\kappa$ . In order to give the proof, we need to use uniform and regular ultrafilters, as well as nonstandard elements; in particular, we shall introduce some related principles which may have independent

interest and which, in a sense, measure “how hard it is” to exclude the uniformity of some ultrafilter, on one hand, or to omit the existence of a nonstandard element in some elementary extension on the other hand. A large part of our methods work for arbitrary regular cardinals in place of  $\omega$ ; in particular, at a certain point, we shall make good use of a notion whose importance has been hinted in Chang [4] and which we call here being “ $\mu$ -nonstandard”; in the particular case  $\mu = \omega$  we get back the classical notion. These techniques allow us to provide a characterization of the compactness properties of products of (possibly uncountable) regular cardinals with the order topology. This seems to have some interest since, as far as we know, all previously known results of this kind have dealt with cardinals endowed with the discrete topology (of course, the two situations coincide in the case of  $\omega$ ).

The following theorem has been proved in [14] in the case when  $\lambda$  is regular, with a slightly simpler condition in place of (3) below. We shall prove the theorem here for arbitrary  $\lambda$ . All the relevant definitions shall be given shortly after the statement.

**Theorem 1** *The space  $\omega^\lambda$  is finally  $\lambda$ -compact if and only if  $\mathcal{L}_{\omega_1, \omega}$  is  $(\lambda, \lambda)$ -compact. More generally, if  $\kappa \geq \lambda$  then the following conditions are equivalent.*

- (1)  $\omega^\kappa$  is  $[\lambda, \lambda]$ -compact.
- (2) The language  $\mathcal{L}_{\omega_1, \omega}$  is  $\kappa$ -( $\lambda, \lambda$ )-compact.
- (3)  $(\lambda, \lambda) \not\equiv^{\kappa} \omega$ .

Unexplained notions and notation are standard; see, e. g., Chang and Keisler [5], Comfort and Negrepolis [6] and Jech [7]. Throughout,  $\lambda, \mu, \kappa$  and  $\nu$  are infinite cardinals,  $\alpha, \beta$  and  $\gamma$  are ordinals,  $X$  is a topological space and  $D$  is an ultrafilter. A cardinal  $\mu$  is also considered as a topological space endowed either with the order topology, or with the coarser *initial interval topology* iit, the topology consisting of the intervals of the kind  $[0, \alpha)$  with  $\alpha \leq \mu$ . No separation axioms are assumed throughout. Products of topological spaces are always assigned the Tychonoff topology. A topological space  $X$  is  $[\mu, \lambda]$ -compact if every open cover of  $X$  by at most  $\lambda$  sets has a subcover by less than  $\mu$  sets. Final  $\kappa$ -compactness is equivalent to  $[\nu, \nu]$ -compactness for every  $\nu \geq \kappa$ . More generally,  $[\mu, \lambda]$ -compactness is equivalent to  $[\nu, \nu]$ -compactness for every  $\nu$  such that  $\mu \leq \nu \leq \lambda$ . The *infinitary language*  $\mathcal{L}_{\omega_1, \omega}$  is like first-order logic, except that we allow conjunctions and disjunctions of countably many formulas. If  $\Sigma$  and  $\Gamma$  are sets of sentences of  $\mathcal{L}_{\omega_1, \omega}$  we say that  $\Gamma$  is  $\mu$ -satisfiable relative to  $\Sigma$  if  $\Sigma \cup \Gamma'$  is satisfiable, for every  $\Gamma' \subseteq \Gamma$  of cardinality  $< \mu$ . If  $\mu \leq \lambda$  we say that  $\mathcal{L}_{\omega_1, \omega}$  is  $\kappa$ -( $\lambda, \mu$ )-compact if  $\Sigma \cup \Gamma$  is satisfiable, whenever  $|\Sigma| \leq \kappa$ ,  $|\Gamma| \leq \lambda$  and  $\Gamma$  is  $\mu$ -satisfiable relative to  $\Sigma$ . We had formerly introduced the notion

of  $\kappa$ -( $\lambda, \mu$ )-compactness for arbitrary logics, extending notions by C. C. Chang, H. J. Keisler, J. A. Makowsky, S. Shelah, A. Tarski, among others. See the book edited by Barwise and Feferman [1], Caicedo [3] and our [14] for references. If  $\kappa \leq \lambda$  then  $\kappa$ -( $\lambda, \mu$ )-compactness reduces to the classical notion of  $(\lambda, \mu)$ -compactness. Notice the reversed order of the cardinal parameters in comparison with the corresponding topological property. If  $D$  is an ultrafilter over some set  $I$  and  $f: I \rightarrow J$  is a function,  $f(D)$  is the ultrafilter over  $J$  defined by  $Y \in f(D)$  if and only if  $f^{-1}(Y) \in D$ . As usual,  $[\lambda]^{<\lambda}$  denotes the set of all subsets of  $\lambda$  of cardinality  $< \lambda$ . We say that an ultrafilter  $D$  over  $[\lambda]^{<\lambda}$  covers  $\lambda$  if  $\{s \in [\lambda]^{<\lambda} \mid \alpha \in s\} \in D$ , for every  $\alpha \in \lambda$ .

**Definition 2** We shall denote by  $(\lambda, \lambda) \not\Rightarrow^\kappa (\mu_\gamma)_{\gamma \in \kappa}$  the following statement.

- (\*) For every sequence of functions  $(f_\gamma)_{\gamma \in \kappa}$  such that  $f_\gamma: [\lambda]^{<\lambda} \rightarrow \mu_\gamma$  for  $\gamma \in \kappa$ , there is some ultrafilter  $D$  over  $[\lambda]^{<\lambda}$  covering  $\lambda$  such that for no  $\gamma \in \kappa$   $f_\gamma(D)$  is uniform over  $\mu_\gamma$ .

We shall write  $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$  when all the  $\mu_\gamma$ 's in (\*) are equal to  $\mu$ .

The negation of  $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$  is denoted by  $(\lambda, \lambda) \Rightarrow^\kappa \mu$ ; similarly for  $(\lambda, \lambda) \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ .

If  $D$  is an ultrafilter over some set  $I$ , a point  $x \in X$  is said to be a  $D$ -limit point of a sequence  $(x_i)_{i \in I}$  of elements of  $X$  if  $\{i \in I \mid x_i \in U\} \in D$ , for every open neighborhood  $U$  of  $x$ . To avoid complex expressions in subscripts, we sometimes shall denote a sequence  $(x_i)_{i \in I}$  as  $\langle x_i \mid i \in I \rangle$ . The next theorem follows easily from Caicedo [3, Section 3], which extended, generalized and simplified former results by A. R. Bernstein, J. Ginsburg and V. Saks, among others. A detailed proof in even more general contexts can be found in Lipparini [11, Proposition 32 (1)  $\Leftrightarrow$  (7)], taking  $\mathcal{F}$  there to be the set of all singletons of  $X$ , and in [12, Theorem 2.3], taking  $\lambda = 1$  there.

**Theorem 3** A topological space  $X$  is  $[\lambda, \lambda]$ -compact if and only if for every sequence  $\langle x_s \mid s \in [\lambda]^{<\lambda} \rangle$  of elements of  $X$  there exists some ultrafilter  $D$  over  $[\lambda]^{<\lambda}$  such that  $D$  covers  $\lambda$  and the sequence has some  $D$ -limit point in  $X$ .

**Corollary 4** Suppose that  $(\mu_\gamma)_{\gamma \in \kappa}$  is a sequence of regular cardinals and that each  $\mu_\gamma$  is endowed either with the order topology or with the  $\text{iit}$  topology. Then  $\prod_{\gamma \in \kappa} \mu_\gamma$  is  $[\lambda, \lambda]$ -compact if and only if  $(\lambda, \lambda) \not\Rightarrow^\kappa (\mu_\gamma)_{\gamma \in \kappa}$ .

**Proof** Let  $X = \prod_{\gamma \in \kappa} \mu_\gamma$  and, for  $\gamma \in \kappa$ , let  $\pi_\gamma: X \rightarrow \mu_\gamma$  be the natural projection. A sequence of functions as in the first line of (\*) in Definition 2 can be naturally identified with a sequence  $\langle x_s \mid s \in [\lambda]^{<\lambda} \rangle$  of elements of  $X$ , by posing  $\pi_\gamma(x_s) = f_\gamma(s)$ .

By Theorem 3,  $X$  is  $[\lambda, \lambda]$ -compact if and only if, for every sequence  $\langle x_s \mid s \in [\lambda]^{<\lambda}$  of elements of  $X$ , there is an ultrafilter  $D$  over  $[\lambda]^{<\lambda}$  covering  $\lambda$  and such that  $\langle x_s \mid s \in [\lambda]^{<\lambda}$  has a  $D$ -limit point in  $X$ . Since a sequence in a product has a  $D$ -limit point if and only if each component has a  $D$ -limit point, the above condition holds if and only if, for each  $\gamma \in \kappa$ ,  $\langle \pi_\gamma(x_s) \mid s \in [\lambda]^{<\lambda}$  has a  $D$ -limit point in  $\mu_\gamma$ , and this happens if and only if, for each  $\gamma \in \kappa$ , there is  $\delta_\gamma \in \mu_\gamma$  such that  $\{s \in [\lambda]^{<\lambda} \mid \pi_\gamma(x_s) < \delta_\gamma\} \in D$ —no matter whether  $\mu_\gamma$  has the order or the  $\text{iit}$  topology. Under the mentioned identification, and since every  $\mu_\gamma$  is regular, this means exactly that each  $f_\gamma(D)$  fails to be uniform over  $\mu_\gamma$ .  $\square$

**Definition 5** We now need to consider a model  $\mathfrak{A}(\lambda, \mu)$  which contains both a copy of  $\langle [\lambda]^{<\lambda}, \subseteq, \{\alpha\} \rangle_{\alpha \in \lambda}$  and a copy of  $\langle \mu, <, \beta \rangle_{\beta \in \mu}$ , where the  $\{\alpha\}$ 's and the  $\beta$ 's are interpreted as constants. Though probably the most elegant way to accomplish this is by means of a two-sorted model, we do not want to introduce technicalities and simply assume that  $A = [\lambda]^{<\lambda} \cup \mu$  and that  $[\lambda]^{<\lambda}$  and  $\mu$  are interpreted, respectively, by unary predicates  $U$  and  $V$ . In details, we let  $\mathfrak{A}(\lambda, \mu) = \langle A, U, V, \subseteq, <, \{\alpha\}, \beta \rangle_{\alpha \in \lambda, \beta \in \mu}$ , where  $U(s)$  holds in  $\mathfrak{A}(\lambda, \mu)$  if and only if  $s \in [\lambda]^{<\lambda}$  and  $V(c)$  holds in  $\mathfrak{A}(\lambda, \mu)$  if and only if  $c \in \mu$ . By abuse of notation we shall not distinguish between symbols and their interpretations. If  $\mathfrak{A}$  is an expansion of  $\mathfrak{A}(\lambda, \mu)$  and  $\mathfrak{B} \equiv \mathfrak{A}$  (that is,  $\mathfrak{B}$  is *elementarily equivalent* to  $\mathfrak{A}$ ), we say that  $b \in B$  *covers*  $\lambda$  if  $U(b)$  and  $\{\alpha\} \subseteq b$  hold in  $\mathfrak{B}$ , for every  $\alpha \in \lambda$ . We say that  $c \in B$  is  $\mu$ -*nonstandard* if  $V(c)$  and  $\beta < c$  hold in  $\mathfrak{B}$ , for every  $\beta \in \mu$ . Of course, in the case  $\mu = \omega$ , we get the usual notion of a nonstandard element. Notice that if  $D$  is an ultrafilter over  $[\lambda]^{<\lambda}$  then  $D$  covers  $\lambda$  in the ultrafilter sense (cf. the sentence immediately before Definition 2) if and only if the  $D$ -class  $[Id]_D$  of the identity on  $[\lambda]^{<\lambda}$  in the ultrapower  $\prod_D \mathfrak{A}(\lambda, \mu)$  covers  $\lambda$  in the present sense. Moreover, if  $\mu$  is regular, then an ultrafilter  $D$  over  $\mu$  is uniform if and only if  $\prod_D \mathfrak{A}(\lambda, \mu)$  has a  $\mu$ -nonstandard element.

**Theorem 6** *If  $\kappa \geq \sup\{\lambda, \mu\}$  then  $(\lambda, \lambda) \not\equiv^\kappa \mu$  if and only if for every expansion  $\mathfrak{A}$  of  $\mathfrak{A}(\lambda, \mu)$  with at most  $\kappa$  new symbols (equivalently, symbols and sorts), there is  $\mathfrak{B} \equiv \mathfrak{A}$  such that  $\mathfrak{B}$  has an element covering  $\lambda$  but no  $\mu$ -nonstandard element.*

**Proof** Suppose that  $(\lambda, \lambda) \not\equiv^\kappa \mu$  and let  $\mathfrak{A}$  be an expansion of  $\mathfrak{A}(\lambda, \mu)$  with at most  $\kappa$  new symbols and sorts. Without loss of generality we may assume that  $\mathfrak{A}$  has Skolem functions, since this adds at most  $\kappa \geq \sup\{\lambda, \mu\}$  new symbols. Enumerate as  $(f_\gamma)_{\gamma \in \kappa}$  all the functions from  $[\lambda]^{<\lambda}$  to  $\mu$  which are definable in  $\mathfrak{A}$  (repeat occurrences, if necessary), and let  $D$  be the ultrafilter given by  $(\lambda, \lambda) \not\equiv^\kappa \mu$ . Let  $\mathfrak{C}$  be the ultrapower  $\prod_D \mathfrak{A}$ . By the remark before the statement of the theorem,  $b = [Id]_D$  is an element in

$\mathfrak{C}$  which covers  $\lambda$ . Let  $\mathfrak{B}$  be the Skolem hull of  $\{b\}$  in  $\mathfrak{C}$ ; thus  $\mathfrak{B} \equiv \mathfrak{C} = \prod_D \mathfrak{A} \equiv \mathfrak{A}$ , and  $b$  covers  $\lambda$  in  $\mathfrak{B}$ . Had  $\mathfrak{B}$  a  $\mu$ -nonstandard element  $c$ , there would be  $\gamma \in \kappa$  such that  $c = f_\gamma(b)$ , by the definition of  $\mathfrak{B}$ . Thus  $c = f_\gamma([Id]_D) = [f_\gamma]_D$ , but this would imply that  $f_\gamma(D)$  is uniform over  $\mu$ , since  $\mu$  is regular, contradicting the choice of  $D$ .

For the converse, suppose that  $(f_\gamma)_{\gamma \in \kappa}$  is a sequence of functions from  $[\lambda]^{<\lambda}$  to  $\mu$ . Let  $\mathfrak{A}$  be the expansion of  $\mathfrak{A}(\lambda, \mu)$  obtained by adding the  $f_\gamma$ 's as unary functions. Notice that we have no need to introduce new sorts. By assumption, there is  $\mathfrak{B} \equiv \mathfrak{A}$  with an element  $b$  covering  $\lambda$  but without  $\mu$ -nonstandard elements. For every formula  $\varphi(y)$  in the vocabulary of  $\mathfrak{A}$  and with exactly one free variable  $y$ , let  $Z_\varphi = \{s \in [\lambda]^{<\lambda} \mid \varphi(s) \text{ holds in } \mathfrak{A}\}$ . Put  $E = \{Z_\varphi \mid \varphi \text{ is as above and } \varphi(b) \text{ holds in } \mathfrak{B}\}$ .  $E$  has trivially the finite intersection property, thus it can be extended to some ultrafilter  $D$  over  $[\lambda]^{<\lambda}$ . For each  $\alpha \in \lambda$ , considering the formula  $\varphi_\alpha: \{\alpha\} \subseteq y$ , we get that  $Z_{\varphi_\alpha} = \{s \in [\lambda]^{<\lambda} \mid \alpha \in s\} \in E \subseteq D$ , thus  $D$  covers  $\lambda$ . Let  $\gamma \in \kappa$ . Since  $\mathfrak{B}$  has no  $\mu$ -nonstandard element, there is  $\beta < \mu$  such that  $f_\gamma(b) < \beta$  holds in  $\mathfrak{B}$ . Letting  $\varphi_\gamma(y)$  be  $f_\gamma(y) < \beta$ , we get that  $Z_{\varphi_\gamma} = \{s \in [\lambda]^{<\lambda} \mid f_\gamma(s) < \beta\} \in E \subseteq D$ , thus  $[0, \beta) \in f_\gamma(D)$ , proving that  $f_\gamma(D)$  is not uniform over  $\mu$ .  $\square$

Theorem 6 explains the reason why we have used the negation of an implication sign in the notation  $(\lambda, \lambda) \not\equiv^\kappa \mu$ . The principle asserts that, modulo possible expansions, the existence of an element covering  $\lambda$  does *not* necessarily *imply* the existence of a  $\mu$ -nonstandard element. Similarly,  $(\lambda, \lambda) \not\equiv^\kappa \mu$  is equivalent to the statement that  $[\lambda, \lambda]$ -compactness of a product of  $\kappa$ -many topological spaces does not imply  $[\mu, \mu]$ -compactness of a factor. See Proposition 7.

**Proof of Theorem 1** The first statement is immediate from the case  $\kappa = \lambda$  of (1)  $\Leftrightarrow$  (2), since  $\omega^\lambda$  is finally  $\lambda^+$ -compact, having a base of cardinality  $\lambda$ . The equivalence of (1) and (3) is the particular case of Corollary 4 when all  $\mu_\gamma$ 's equal  $\omega$ . Thus, in view of Theorem 6, it is enough to prove that (2) is equivalent to the necessary and sufficient condition given there for  $(\lambda, \lambda) \not\equiv^\kappa \omega$ . For the simpler direction, suppose that  $\mathcal{L}_{\omega_1, \omega}$  is  $\kappa$ -( $\lambda, \lambda$ )-compact and that  $\mathfrak{A}$  is an expansion of  $\mathfrak{A}(\lambda, \omega)$  with at most  $\kappa$  new symbols. Let  $\Sigma$  be the elementary (first order) theory of  $\mathfrak{A}$  plus an  $\mathcal{L}_{\omega_1, \omega}$  sentence asserting that there exists no nonstandard element and let  $\Gamma = \{\{\alpha\} \subseteq b \mid \alpha \in \lambda\}$ . By applying  $\kappa$ -( $\lambda, \lambda$ )-compactness of  $\mathcal{L}_{\omega_1, \omega}$  to the above sets of sentences we get a model  $\mathfrak{B}$  as requested by the condition in Theorem 6.

The reverse direction is a variation on a standard reduction argument. Suppose that the condition in Theorem 6 holds. If  $\mathfrak{A}$  is a many-sorted expansion of the model  $\mathfrak{A}(\lambda, \omega)$  introduced in Definition 5 then, for every  $\mathfrak{B} \equiv \mathfrak{A}$  such that  $\mathfrak{B}$  has no nonstandard

element, a formula  $\psi$  of  $\mathcal{L}_{\omega_1, \omega}$  of the form  $\bigwedge_{n \in \omega} \varphi_n(\bar{x})$  is equivalent to  $\forall y (V(y) \Rightarrow R_\psi(y, \bar{x}))$  in some expansion  $\mathfrak{B}^+$  of  $\mathfrak{B}$  with a newly introduced relation  $R_\psi$  such that  $\forall \bar{x} (R_\psi(n, \bar{x}) \Leftrightarrow \varphi_n(\bar{x}))$  holds in  $\mathfrak{B}^+$ , for every  $n \in \omega$ . Here we are using in an essential way the fact that in a sentence of  $\mathcal{L}_{\omega_1, \omega}$  we can quantify away only a finite number of variables, hence we can do with a finitary relation  $R_\psi$ . Thus, given  $\Sigma$  and  $\Gamma$  sets of sentences as in the definition of  $\kappa$ -( $\lambda, \lambda$ )-compactness, assuming that we work in models without nonstandard elements, iterating the above procedure for all subformulas of the sentences under consideration and working in some appropriately expanded vocabulary, we may reduce all the relevant satisfaction conditions to the case in which  $\Sigma$  and  $\Gamma$  are sets of first order sentences. In the situation at hand we need to add to  $\Sigma$  all the sentences of the form  $\forall \bar{x} (R_\psi(n, \bar{x}) \Leftrightarrow \varphi_n(\bar{x}))$  as above, but easy computations show that we still have  $|\Sigma| \leq \kappa$ , since  $\kappa \geq \lambda$ . If  $\Gamma = \{\gamma_\alpha \mid \alpha \in \lambda\}$  is  $\lambda$ -satisfiable relative to  $\Sigma$ , construct a many-sorted expansion  $\mathfrak{A}$  of  $\mathfrak{A}(\lambda, \omega)$  with a further new binary relation  $S$  such that, for every  $s \in [\lambda]^{<\lambda}$ ,  $\{z \in A \mid S(s, z)\}$  models  $\Sigma \cup \{\gamma_\alpha \mid \alpha \in s\}$ . This is possible, since  $\Gamma$  is  $\lambda$ -satisfiable relative to  $\Sigma$ . If  $\mathfrak{B} \equiv \mathfrak{A}$  is given by  $(\lambda, \lambda) \not\equiv^\kappa \omega$  and  $b \in B$  covers  $\lambda$  then  $\{z \in B \mid S(b, z)\}$  models  $\Sigma \cup \Gamma$ . Indeed, for every  $\alpha \in \lambda$  the sentence  $\forall w (\{\alpha\} \subseteq w \wedge U(w) \Rightarrow \gamma_\alpha^{S(w, -)})$  is satisfied in  $\mathfrak{A}$  hence, by elementarity, it is satisfied in  $\mathfrak{B}$ , too, where by  $\gamma_\alpha^{S(w, -)}$  we denote a *relativization* of  $\gamma_\alpha$  to  $S(w, -)$ , that is, a sentence such that if  $\mathfrak{C}$  is a model,  $c \in C$  and  $\{z \in C \mid S(c, z)\}$  is itself a model for an appropriate vocabulary, then  $\gamma_\alpha^{S(c, -)}$  is satisfied in  $\mathfrak{C}$  if and only if  $\gamma_\alpha$  is satisfied in  $\{z \in C \mid S(c, z)\}$ . Similarly, for every  $\sigma \in \Sigma$ ,  $\forall w (U(w) \Rightarrow \sigma^{S(w, -)})$  is satisfied in  $\mathfrak{A}$  hence it is satisfied in  $\mathfrak{B}$ . See the book edited by Barwise and Feferman [1] for further technical details, in particular about relativization and about dealing with constants. Notice also that in the above proof we do need the many-sorted (or relativized) version of the condition in Theorem 6, since when  $\kappa$  is large the models witnessing the  $\lambda$ -satisfiability of  $\Gamma$  relative to  $\Sigma$  might have cardinality exceeding the cardinality of the base set of  $\mathfrak{A}(\lambda, \omega)$ .  $\square$

In [14] we have proved results similar to those presented here, but we have dealt mostly with regular  $\lambda$ . In that situation we could do with a principle simpler than  $(\lambda, \lambda) \not\equiv^\kappa (\mu_\gamma)_{\gamma \in \kappa}$ ; we have denoted that principle by  $\lambda \not\equiv (\mu_\gamma)_{\gamma \in \kappa}$ . The definition is essentially as Definition 2, except for the fact that this time we use uniform ultrafilters over  $\lambda$  rather than ultrafilters over  $[\lambda]^{<\lambda}$  covering  $\lambda$ . For  $\lambda$  regular the two principles are easily seen to be equivalent. Indeed, if  $\lambda$  is regular,  $f_1: [\lambda]^{<\lambda} \rightarrow \lambda$  is defined by  $f_1(s) = \sup s$ , and an ultrafilter  $D$  over  $[\lambda]^{<\lambda}$  covers  $\lambda$  then  $f_1(D)$  is uniform over  $\lambda$ . The assumption that  $\lambda$  is regular is needed to get that the range of  $f_1$  is contained in  $\lambda$ . On the other hand, the function  $f_2: \lambda \rightarrow [\lambda]^{<\lambda}$  which assigns to  $\alpha \in \lambda$  the set  $\{\beta \in \lambda \mid \beta < \alpha\}$  is such that if an ultrafilter  $D$  over  $\lambda$  is uniform then  $f_2(D)$  over



$[\lambda]^{<\lambda}$  covers  $\lambda$ . Of course, under the standard nowadays usual identification of an ordinal with the set of all smaller ordinals,  $f_2(\alpha) = \alpha$ . Using  $f_1$  and  $f_2$  as tools for transferring from  $[\lambda]^{<\lambda}$  to  $\lambda$  and conversely, one immediately sees that  $\lambda \not\Rightarrow^{\kappa} (\mu_\gamma)_{\gamma \in \kappa}$  implies  $(\lambda, \lambda) \not\Rightarrow^{\kappa} (\mu_\gamma)_{\gamma \in \kappa}$  for every  $\lambda$ , and that the two principles are equivalent for  $\lambda$  regular.

The above principles are interesting only for small values of  $\kappa$ . Indeed if  $\kappa \geq \mu^{\lambda < \lambda}$  then  $(\lambda, \lambda) \not\Rightarrow^{\kappa} \mu$  is equivalent to the statement that there exists an ultrafilter over  $[\lambda]^{<\lambda}$  covering  $\lambda$  such that for no function  $f: [\lambda]^{<\lambda} \rightarrow \mu$   $f(D)$  is uniform over  $\mu$ . This is because there are exactly  $\mu^{\lambda < \lambda}$  functions from  $[\lambda]^{<\lambda}$  to  $\mu$ . Thus while in general we obtain a stronger statement when we increase  $\kappa$  in  $(\lambda, \lambda) \not\Rightarrow^{\kappa} \mu$ , at the point  $\kappa = \mu^{\lambda < \lambda}$  we have already reached the strongest possible notion. Recall that an ultrafilter  $D$  over a set  $I$  is  $(\lambda, \lambda)$ -regular if there is a function  $f: I \rightarrow [\lambda]^{<\lambda}$  such that  $f(D)$  covers  $\lambda$ ; and that  $D$  is  $\mu$ -decomposable if there is a function  $f: I \rightarrow \mu$  such that  $f(D)$  is uniform over  $\mu$ . Hence for  $\kappa \geq \mu^{\lambda < \lambda}$  we get that  $(\lambda, \lambda) \not\Rightarrow^{\kappa} \mu$  is equivalent to the statement that there is a  $(\lambda, \lambda)$ -regular not  $\mu$ -decomposable ultrafilter. The problem of the existence of such ultrafilters is connected with difficult set-theoretical problems involving large cardinals, forcing and pcf-theory, and has been widely studied, sometimes in equivalent formulations. See [10] for more information. In a couple of papers we have somewhat attempted a study of the more comprehensive (hence more difficult!) relation  $(\lambda, \lambda) \not\Rightarrow^{\kappa} \mu$ . References can be found in [10, 14]. Roughly, while for large  $\kappa$  we get notions related to measurability, on the other hand, for smaller values of  $\kappa$  we get corresponding variants of weak compactness, as the present note itself exemplifies. Notice that in some previous works we had given the definition of, say,  $(\lambda, \lambda) \not\Rightarrow^{\kappa} \mu$  by means of the equivalent condition given here by Theorem 6, in which the assumption  $\kappa \geq \sup\{\lambda, \mu\}$  is made. Hence in some places the notation we have used might be not consistent with the present one (but only when small values of  $\kappa$  are taken into account).

Finally, expanding a remark we have presented in [14], we notice that, though we have stated our topological results in terms of products of cardinals, they can be reformulated in a way that involves arbitrary products of topological spaces.

**Proposition 7** *If  $(\mu_\gamma)_{\gamma \in \kappa}$  is a sequence of regular cardinals then the following conditions are equivalent.*

- (1)  $\prod_{\gamma \in \kappa} \mu_\gamma$  is not  $[\lambda, \lambda]$ -compact, where each  $\mu_\gamma$  is equivalently endowed either with the order topology or with the  $\text{iit}$  topology.
- (2)  $(\lambda, \lambda) \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$

- (3) For every set  $I$  and every product  $X = \prod_{i \in I} X_i$  of topological spaces, if  $X$  is  $[\lambda, \lambda]$ -compact, then for every injective function  $g: \kappa \rightarrow I$  there is  $\gamma \in \kappa$  such that  $X_{g(\gamma)}$  is  $[\mu_\gamma, \mu_\gamma]$ -compact.

**Proof** (1)  $\Leftrightarrow$  (2) is Corollary 4 in contrapositive form. (3)  $\Rightarrow$  (1) is trivial by taking  $I = \kappa$  and  $g$  to be the identity function and observing that  $\mu_\gamma$  is not  $[\mu_\gamma, \mu_\gamma]$ -compact (with either topology), since  $\mu_\gamma$  is regular. To finish the proof we shall prove that if (3) fails then (1) fails. So suppose that (3) fails as witnessed by some  $[\lambda, \lambda]$ -compact  $X = \prod_{i \in I} X_i$  and an injective  $g: \kappa \rightarrow I$  such that no  $X_{g(\gamma)}$  is  $[\mu_\gamma, \mu_\gamma]$ -compact. It is easy to see that if  $\mu$  is regular then a space is not  $[\mu, \mu]$ -compact if and only if there is a continuous surjective function  $h: X \rightarrow \mu$  with the iit topology; see, e. g., Lipparini [13, Lemma 4]. Hence for each  $\gamma \in \kappa$  we have a continuous surjective function  $h_\gamma: X_\gamma \rightarrow \mu_\gamma$  and, by naturality of products and since  $g$  is injective, a continuous surjective  $h: X \rightarrow \prod_{\gamma \in \kappa} \mu_\gamma$ . Thus if  $X$  is  $[\lambda, \lambda]$ -compact then so is  $\prod_{\gamma \in \kappa} \mu_\gamma$ , since  $[\lambda, \lambda]$ -compactness is preserved under surjective continuous images, hence (1) fails in the case the  $\mu_\gamma$ 's are assigned the iit topology. This is enough, since we have already proved that in (1) we can equivalently consider either topology, since in each case (1) is equivalent to (2).  $\square$

Notice that, in particular, it follows from Proposition 7 that if  $\mu$  is regular,  $(\lambda, \lambda) \Rightarrow^\kappa \mu$  and some product is  $[\lambda, \lambda]$ -compact, then all but at most  $< \kappa$  factors are  $[\mu, \mu]$ -compact. In this way we obtain alternative proofs—as well as various strengthenings—of many of the results we have proved in [9].

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